Appendix A
Nonrigid affine correction

One way to estimate a correction matrix $\mathbf{J} \doteq \mathbf{M}\backslash\tilde{\mathbf{M}}$ generalizes the solution for the rigid affine correction given above. The strategy is to break $\mathbf{M}$ into column-triples. Each column-triple is a stack of rotation matrices scaled by morph weights. Let $\mathbf{m}_{f_{k,x}}^{\top}, \mathbf{m}_{f_{k,y}}^{\top} \in \mathbf{M}$ be the x and y projections in frame f as given by column-triple k. As in the rigid affine correction, in a properly structured motion matrix $\mathbf{M}$ these vectors should have equal norm and be orthogonal:

$$\forall_{f,k} \left[\|\mathbf{m}_{f_{k,x}}\| = \|\mathbf{m}_{f_{k,y}}\|\right] \wedge \left[\mathbf{m}_{f_{k,x}}^{\top}\mathbf{m}_{f_{k,y}} = 0\right]. \tag{1}$$

Morever, their projections onto vectors from other column triples should also have equal norm (because all column-triples have the same rotations):

$$\forall_{f,k,j} \left[\mathbf{m}_{f_{k,x}}\mathbf{m}_{f_{j,x}} = \mathbf{m}_{f_{k,y}}\mathbf{m}_{f_{j,y}}\right] \wedge \left[\mathbf{m}_{f_{k,x}}^{\top}\mathbf{m}_{f_{j,y}} = 0\right]. \tag{2}$$

This yields a system of equations

$$\forall_{f,k,j}\ (\mathrm{vec}(\mathbf{m}_{f_{k,x}}\mathbf{m}_{f_{j,x}}^{\top} - \mathbf{m}_{f_{k,y}}\mathbf{m}_{f_{j,y}}^{\top}))^{\top}\, \mathrm{vec}\, \mathbf{H}_{k,j} = 0, \tag{3}$$

$$\forall_{f,k,j}\ (\mathrm{vec}(\mathbf{m}_{f_{k,x}}\mathbf{m}_{f_{j,y}}^{\top}))^{\top}\, \mathrm{vec}\, \mathbf{H}_{k,j} = 0. \tag{4}$$

Now recall that each $\mathbf{H}_{k,j}$ is the outer product of two column-triples in $(\mathbf{J}^{-1})$, e.g.,

$$\mathbf{H}_{k,j} = (\mathbf{J}^{-1})_{\mathrm{cols}(3k-2,3k-1,3k)}(\mathbf{J}^{-1})_{\mathrm{cols}(3j-2,3j-1,3j)}^{\top}. \tag{5}$$

Consequently, the matrix

$$\mathbf{H} \doteq \begin{bmatrix} \mathbf{H}_{1,1} & \cdots & \mathbf{H}_{1,K} \\ \vdots & \ddots & \vdots \\ \mathbf{H}_{K,1} & \cdots & \mathbf{H}_{K,K} \end{bmatrix} = (\mathbf{J}^{-1})^{(3K,3)}(\mathbf{J}^{-1})^{(3K,3)\top} \tag{6}$$

should be symmetric with rank 3. Let $\mathbf{V}\mathbf{\Lambda}\mathbf{V}^{\top} \overset{\mathrm{EIG}_3}{\longleftarrow} \mathbf{H}$ be a truncated decomposition of $\mathbf{H}$ using its three largest eigenvalues and their associated eigenvectors. Then the desired correction is $(\mathbf{J}^{-1}) = (\mathbf{V}\sqrt{\mathbf{\Lambda}})^{(3K,3)}$.

Although formally "correct," this procedure is of limited use because in order to express eqns. (3–4) in terms of $\mathbf{J}^{-1}$ we must make the substitution $\mathbf{m}_{f_{k,x}}^{\top} \to \tilde{\mathbf{m}}_{f_x}^{\top}(\mathbf{J}^{-1})_{\mathrm{cols}(3k-2,3k-1,3k)}$, which makes the constraints on all $\mathbf{H}_{k,j}$ nearly identical. Consequently the linear system is rank-deficient, because the number of unknowns in $\mathbf{H}$ grows as $O(K^4)$ (or $O(K^3)$ if one only considers $j = \{k, k+1\}$) while the number of true unknowns in $\mathbf{J}^{-1}$ grows as $O(K^2)$. In practice, there are enough constraints to support a usable estimate of the first three columns of $\mathbf{J}^{-1}$. We can therefore calculate the first column-triple of $\hat{\mathbf{M}}$, project $\tilde{\mathbf{M}}$ into the $3K-3$ dimensional space orthogonal to this, and repeat the procedure to get the next column triple of $\hat{\mathbf{M}}$. A generalized SVD solution for factoring $\mathbf{H}$ without explicitly computing its elements (thereby avoiding the rank-deficient division) requires some extra pages to explain and therefore will be published separately.